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A MODEL IN WHICH THERE ARE JECH–KUNEN TREES BUT THERE ARE NO KUREPA TREES

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ABSTRACT

By an ω_1-tree we mean a tree of power ω_1 and height ω_1. We call an ω_1-tree a Jech–Kunen tree if it has κ–many branches for some κ strictly between ω_1 and 2^{ω_1}. In this paper we construct the models of CH plus $2^{\omega_1} > \omega_2$, in which there are Jech–Kunen trees and there are no Kurepa trees.

An partially ordered set, or poset for short, $\langle T, <_T\rangle$ is called a tree if for every $t \in T$ the set $\{s \in T\colon s <_T t\}$ is well-ordered under $<_T$. The order type of that set is called the height of t in T, denoted by $\mathrm{ht}_T(t)$. We will not distinguish a tree from its base set. For every ordinal α, let T_α, the α-th level

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of T, $= \{t \in T\colon \mathrm{ht}_T(t) = \alpha\}$ and $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$. Let $\mathrm{ht}(T)$, the height of T, is the smallest ordinal α such that $T_\alpha = \emptyset$. By a **branch** of T we mean a linearly ordered subset of T which intersects every nonempty level of T. Let $\mathcal{B}(T)$ be the set of all branches of T. T' is called a **subtree** of T if $T' \subseteq T$ and $<_{T'} = <_T \bigcap T' \times T'$ (T' inherits the order of T).

T is called an ω_1-tree if $|T| = \omega_1$ and $\mathrm{ht}(T) = \omega_1$. An ω_1-tree T is called a **Kurepa tree** if $|\mathcal{B}(T)| > \omega_1$ and for every $\alpha \in \omega_1$, $|T_\alpha| < \omega_1$. An ω_1-tree is called a **Jech–Kunen tree** if $\omega_1 < |\mathcal{B}(T)| < 2^{\omega_1}$.

The independence of the existence of Kurepa trees was proved by J. H. Silver (see [K2, §3 of Chapter VIII]). T. Jech in [Je1] constructed by forcing a model of CH plus $2^{\omega_1} > \omega_2$, in which there is a Jech–Kunen tree. In fact, it is a Kurepa tree with fewer than 2^{ω_1}-many branches. The independence of the existence of Jech–Kunen trees under CH plus $2^{\omega_1} > \omega_2$ was given by K. Kunen [K1]. In his paper he gave an equivalent form of Jech–Kunen trees in terms of compact Hausdorff spaces. The detailed proof can be found in [Ju, Theorem 4.8].

In both Silver and Kunen's proofs, the existence of a strongly inaccessible cardinal was assumed (the assumption is also necessary). The technique they used to kill all Kurepa trees or Jech–Kunen trees is to show that if an ω_1-tree T has a new branch in an ω_1-closed forcing extension, then T must have a subtree which is isomorphic to $\langle 2^{<\omega_1}, \subseteq\rangle$, a complete binary tree of height ω_1. So in Kunen's model not only all Jech–Kunen trees are killed, but also all Kurepa trees are killed.

R. Jin in [Ji1] started discussing the differences between Kurepa trees and Jech–Kunen trees. He showed that it is independent of CH plus $2^{\omega_1} > \omega_2$ that there exists a Kurepa tree which has no Jech–Kunen subtrees. He also showed that it is independent of CH plus $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree which has no Kurepa subtrees. In his proofs some strongly inaccessible cardinals were assumed and later, Kunen eliminated the large cardinal assumption for one of the proofs.

In [Ji2] Jin proved that assuming the existence of two inaccessible cardinals, it is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exist Kurepa trees and there are no Jech–Kunen trees.

The problem whether CH plus $2^{\omega_1} > \omega_2$ is consistent with that there exist Jech–Kunen trees and there are no Kurepa trees, was posed in [Ji2]. We will answer the question in this paper by assuming naturally the existence of a strongly inaccessible cardinal.

Before proving our results we need more notations and definitions.

A tree T is called **normal** if,

(1) every $t \in T$ has at least two immediate successors,

(2) for every $t \in T$ and an ordinal α such that $\mathrm{ht}_T(t) < \alpha < \mathrm{ht}(T)$, there exists $t' \in T_\alpha$ such that $t <_T t'$.

A tree $C = \{c_s : s \in 2^{<\omega}\}$ is called a **Cantor tree** if the map $s \mapsto c_s$ is an isomorphism from $\langle 2^{<\omega}, \subseteq\rangle$ to C. For convenience we assume, from now on, that every tree considered in this paper is a subtree of $\langle 2^{<\omega_1}, \subseteq\rangle$ with the unique root $\emptyset$. By that way we can define the least upper bound of an increasing sequence in a tree by taking its union. Let $\lim(\omega_1)$ be the set of all limit ordinals in ω_1. Let T be a tree and $\delta \in \lim(\omega_1)$. A subtree C of T is called **cofinal** in $T \restriction \delta$ if for every $B \in \mathcal{B}(C)$, the set $\{\mathrm{ht}_T(t) : t \in B\}$ is cofinal in δ. T is called **complete at level** δ if for every $B \in \mathcal{B}(T \restriction \delta)$, $\bigcup B \in T_\delta$. T is called **properly pruned at level** δ if for every Cantor subtree $C = \{c_s : s \in 2^{<\omega}\}$ of T which is cofinal in $T \restriction \delta$, there exist $f, g \in 2^\omega$ such that $\bigcup_{n\in\omega} c_{f\restriction n} \in T_\delta$ and $\bigcup_{n\in\omega} c_{g\restriction n} \notin T_\delta$

Let $S \subseteq \lim(\omega_1)$. A tree is called S-**properly pruned** if for every $\alpha \in \lim(\omega_1)$, $\alpha \notin S$ implies that T is complete at level α, and $\alpha \in S$ implies that T is properly pruned at level α.

Let I be an index set and T be a tree. For every $F \in T^I$, let $\mathrm{supt}(F)$, the support of F, be the set $\{i \in I : F(i) \neq \emptyset\}$. Let $F, G \in T^I$. Define $F \preccurlyeq G$ iff for every $i \in I$, $F(i) \leq G(i)$. We call $F \in T^I$ **uniform** at δ for some $\delta \in \omega_1$ if for every $i \in \mathrm{supt}(F)$, $\mathrm{ht}_T(F(i)) = \delta$. Let $C = \{F_s : s \in 2^{<\omega}\} \subseteq T^I$ be a Cantor tree (under $\preccurlyeq$). C is called **uniformly cofinal** in $(T \restriction \delta)^I$ for some $\delta \in \omega_1$ if for every $s \in 2^{<\omega}$, there is a $\delta_s < \delta$ such that F_s is uniform at δ_s and for every $i \in \bigcup_{s\in 2^{<\omega}} \mathrm{supt}(F_s)$, the subtree $\{F_s(i) : s \in 2^{<\omega}\}$ of T is cofinal in $T \restriction \delta$. We use $\perp$ for the word "incompatible". For example, for any $s, t \in 2^{<\omega}$, $s \perp t$ means $s \bigcup t$ is not a function. For any $F, G \in T^I$, we call that F and G are **completely incompatible** if for any $i \in \mathrm{supt}(F)$ and any $j \in \mathrm{supt}(G)$, $F(i) \perp G(j)$ ($F(i)$ and $G(j)$ have no common upper bound in T). Now C is called **separated** if for any $s, s' \in 2^{<\omega}$, $s \perp s'$ implies that F_s and $F_{s'}$ are completely incompatible.

Let T be a tree and $\delta \in \lim(\omega_1)$. We recall that T is **properly pruned in countable products at level** δ if for every Cantor tree $C = \{F_s : s \in 2^{<\omega}\} \subseteq T^I$, which is separated and uniformly cofinal in $(T \restriction \delta)^\omega$, there exist $f, g \in 2^\omega$ such that for every $i \in \bigcup_{n\in\omega} \mathrm{supt}(F_{f\restriction n})$, $\bigcup_{n\in\omega} F_{f\restriction n}(i) \in T_\delta$ and for every $i \in \bigcup_{n\in\omega} \text{supt}(F_{g\restriction n})$, $\bigcup_{n\in\omega} F_{g\restriction n}(i) \notin T_\delta$.

Let $S \subseteq \lim(\omega_1)$. A tree is called S-**properly pruned in countable products** if for every $\alpha \in \lim(\omega_1)$, $\alpha \notin S$ implies that T is complete at level α, and $\alpha \in S$ implies that T is properly pruned in countable products at level α.

LEMMA 1: *Let T be a tree and I be an index set. For any Cantor tree $C = \{F_s: s \in 2^{<\omega}\} \subseteq T^I$, if C is separated, then for any $f, g \in 2^\omega$, $f \neq g$ implies that $\langle\bigcup_{n\in\omega} F_{f\restriction n}(i)\rangle_{i\in I}$ and $\langle\bigcup_{n\in\omega} F_{g\restriction n}(i)\rangle_{i\in I}$ are completely incompatible.*

Proof: Let $i \in \bigcup_{n\in\omega} \text{supt}(F_{f\restriction n})$ and $j \in \bigcup_{n\in\omega} \text{supt}(F_{g\restriction n})$. Let $m \in \omega$ such that $i \in \text{supt}(F_{f\restriction m})$, $j \in \text{supt}(F_{g\restriction m})$ and $f\restriction m \neq g\restriction m$. Then $\bigcup_{n\in\omega} F_{f\restriction n}(i)$ and $\bigcup_{n\in\omega} F_{g\restriction n}(j)$ are compatible implies that $F_{f\restriction m}(i)$ and $F_{g\restriction m}(j)$ are compatible, a contradiction. ∎

LEMMA 2 (CH): *For any $S \subseteq \lim(\omega_1)$, there exists a normal ω_1-tree which is S-properly pruned in countable products.*

Proof: We construct $T_\delta \subseteq 2^\delta$ recursively on $\delta < \omega_1$ and $T = \bigcup_{\delta<\omega_1} T_\delta$ will be the tree we want.

CASE 1: *$\delta = \beta + 1$ for some $\beta \in \omega_1$.* Let $T_\delta = \{t^\frown\langle l\rangle: t \in T_\beta, l = 0, 1\}$.

CASE 2: *$\delta \in \lim(\omega_1) \setminus S$.* Let $T_\delta = \{\bigcup B: B \in \mathcal{B}(T\restriction\delta)\}$.

CASE 3: *$\delta \in S$.* Let $\mathcal{C}$ be the set of all Cantor trees which are separated and uniformly cofinal in $(T\restriction\delta)^\omega$. By CH we have that $|\mathcal{C}| \leq (\omega_1^\omega)^\omega = \omega_1$. Let $\mathcal{C} = \{C^\alpha: \alpha \in \omega_1\}$ be an enumeration, where $C^\alpha = \{F_s^\alpha: s \in 2^{<\omega}\}$. We now want to find a set $X \subseteq \{\bigcup B: B \in \mathcal{B}(T\restriction\delta)\}$ such that for every $\alpha \in \omega_1$, there are $f, g \in 2^\omega$ such that

$$\{\bigcup_{n\in\omega} F_{f\restriction n}^\alpha(i): i \in \omega\} \subseteq X \bigcup \{\emptyset\}$$

and

$$\{\bigcup_{n\in\omega} F_{g\restriction n}^\alpha(i): i \in \omega\} \bigcap X = \emptyset.$$

If X is found, we let $T_\delta = X$.

We now build X_γ and Y_γ recursively such that,

(1) X_γ and Y_γ are countable,

(2) $\gamma < \gamma' < \omega_1$ implies that $X_\gamma \subseteq X_{\gamma'}$ and $Y_\gamma \subseteq Y_{\gamma'}$,

(3) $X_\gamma \bigcap Y_\gamma = \emptyset$ for every $\gamma \in \omega_1$,

(4) for every $\gamma \in \omega_1$, there exist $f, g \in 2^\omega$ such that $\{\bigcup_{n\in\omega} F^\gamma_{f\restriction n}(i): i \in \omega\} \subseteq X_{\gamma+1}$ and $\{\bigcup_{n\in\omega} F^\gamma_{g\restriction n}(i): i \in \omega\} \subseteq Y_{\gamma+1}$.

Let $X_0 = Y_0 = \emptyset$. Let $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$ and $Y_\gamma = \bigcup_{\beta<\gamma} Y_\beta$ if $\gamma \in \lim(\omega_1)$. For $\gamma + 1$, since X_γ and Y_γ are countable and C^γ is separated, by Lemma 1, there exist $f, g \in 2^\omega, f \neq g$ such that

$$(X_\gamma \bigcup Y_\gamma) \bigcap (\{\bigcup_{n\in\omega} F^\gamma_{f\restriction n}(i): i \in \omega\} \bigcup \{\bigcup_{n\in\omega} F^\gamma_{g\restriction n}(i): i \in \omega\}) = \emptyset.$$

Hence let

$$X_{\gamma+1} = X_\gamma \bigcup \{\bigcup_{n\in\omega} F^\gamma_{f\restriction n}(i): i \in \bigcup_{n\in\omega} \mathrm{supt}(F^\gamma_{f\restriction n})\}$$

and

$$Y_{\gamma+1} = Y_\gamma \bigcup \{\bigcup_{n\in\omega} F^\gamma_{g\restriction n}(i): i \in \bigcup_{n\in\omega} \mathrm{supt}(F^\gamma_{g\restriction n})\}.$$

Then $X = \bigcup_{\gamma\in\omega_1} X_\gamma$ is the set we want. ∎

LEMMA 3: *Let $S \subseteq \lim(\omega_1)$. T is S-properly pruned in countable products implies that T is S-properly pruned.*

Proof: If $C = \{c_s: s \in 2^{<\omega}\} \subseteq T$ is a Cantor tree which is cofinal in $T \restriction \delta$ for some $\delta \in S$, then the Cantor tree $D = \{F_s: s \in 2^{<\omega}\} \subseteq T^\omega$, where $F_s(0) = c_s$ and $F_s(i) = \emptyset$ for every $i \neq 0$, is separated and uniformly cofinal in $(T \restriction \delta)^\omega$. ∎

LEMMA 4: *Let $S \subseteq \lim(\omega_1)$ and T be S-properly pruned in countable products. Let $C = \{F_s: s \in 2^{<\omega}\}$ be a separated and uniformly cofinal Cantor subtree in $(T \restriction \delta)^\omega$ for some $\delta \in S$. Then there are uncountably many $f \in 2^\omega$ such that for every $i \in \bigcup_{n\in\omega} \mathrm{supt}(F_{f\restriction n})$, $\bigcup_{n\in\omega} F_{f\restriction n}(i) \in T_\delta$.*

Proof: Suppose that the lemma is not true. Then we can find a Cantor subtree $C' = \{F'_s: s \in 2^{<\omega}\} \subseteq C$ such that for every $f \in 2^\omega$, there exists $i \in \omega$, $\bigcup_{n\in\omega} F'_{f\restriction n}(i) \notin T_\delta$. Since C' is a subtree of C, C' itself is also separated and uniformly cofinal in $(T \restriction \delta)^\omega$. That contradicts the definition of the S-properly prunedness in countable products. ∎

Next we shall use the forcing method to construct desired models. For the terminology and basic facts of forcing, see [K2] and [Je2]. We always assume the consistency of ZFC and let M be always a countable transitive model of ZFC. In the forcing arguments, we always let $\dot{a}$ be a name of a if a is not in the ground model. For every element a in the ground model, we will not distinguish a from its canonical name.

Let I, J be two sets. Let

$$Fn(I, J, \omega_1) = \{p : p \subseteq I \times J \text{ is a function and } |p| < \omega_1\}$$

be a poset ordered by reverse inclusion. Let I be a subset of a cardinal κ. Let

$$Lv(I, \omega_1) = \{p : p \subseteq (I \times \omega_1) \times \kappa \text{ is a function, } |p| < \omega_1 \\ \text{and } \forall \langle \alpha, \beta \rangle \in \text{dom}(p)(p(\alpha, \beta) \in \alpha)\}$$

be a poset ordered by reverse inclusion. Let T be a tree and I be an index set. Let

$$\mathbb{P}(T, I, \omega_1) = \{F : F \in T^I \text{ and } |\text{supt}(F)| < \omega_1\}.$$

The order of $\mathbb{P}(T, I, \omega_1)$ is defined as the reverse order of T^I, or $F \leq_{\mathbb{P}(T,I,\omega_1)} G$ iff $G \preccurlyeq F$.

LEMMA 5: *Let T be a normal ω_1-tree and I be an index set. For any $p, q \in \mathbb{P}(T, I, \omega_1)$, there exist $p', q' \in \mathbb{P}(T, I, \omega_1)$ such that $p' < p$ and $q' < q$, p', q' are uniform at δ for some $\delta \in \omega_1$, and p' is completely incompatible with q'.*

Proof: Let $\alpha \in \omega_1$ be large enough so that $p, q \in (T \restriction \alpha)^I$ (α exists because p, q both have countable supports). Let $\delta > \alpha$ be countable such that for every $i \in \text{supt}(p)$

$$|\{t \in T_\delta : p(i) <_T t\}| \geq \omega,$$

and for every $j \in \text{supt}(q)$

$$|\{t \in T_\delta : q(j) <_T t\}| \geq \omega.$$

δ exists because T is normal. Let

$$\text{supt}(p) = \{i_n : n \in \omega\}$$

and

$$\text{supt}(q) = \{j_n : n \in \omega\}.$$

We now define $p'(i_n)$ and $q'(j_n)$ such that

$$p'(i_n), q'(j_n) \in T_\delta,$$

$$p'(i_n) > p(i_n),\ q'(j_n) > q(j_n),\ p'(i_n) \neq q'(j_n)$$

and

$$p'(i_n), q'(j_n) \notin \{p'(i_m), q'(j_m): m < n\}.$$

Let $p'(i) = \emptyset$ if $i \notin \operatorname{supt}(p)$ and let $q'(j) = \emptyset$ if $j \notin \operatorname{supt}(q)$. Then p' and q' are the desired elements. ∎

Let $\mathbb{P}$ be a poset and $D \subseteq \mathbb{P}$. D is called **dense** in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is $d \in D$ such that $d \leq p$. D is called **open** in $\mathbb{P}$ if for every $p \in \mathbb{P}$ and $d \in D$, $p \leq d$ implies that $p \in D$. $\mathbb{P}$ is called ω_1-**Baire** if for any countable sequence $\langle D_n: n \in \omega\rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{n\in\omega} D_n$ is dense in $\mathbb{P}$.

LEMMA 6: *In M let $\mathbb{P}$ be a poset which is ω_1-Baire. Let G be a $\mathbb{P}$-generic filter over M. Then $M^\omega \bigcap M[G] \subseteq M$.*

Proof: Let $h \in M[G]$ be a function from ω to A, where $A \in M$. We work in M and let $p \in G$ such that

$$p \Vdash (\dot{h} \text{ is a function from } \omega \text{ to } A).$$

For every $n \in \omega$, let

$$D_n = \{q \in \mathbb{P}: q \perp p \text{ or } \exists a \in A(q \Vdash \dot{h}(n) = a)\}.$$

Then D_n is dense open in $\mathbb{P}$. Let $\overline{p} \in \bigcap_{n\in\omega} D_n$ such that $\overline{p} \leq p$. Then

$$h = \{\langle n, a\rangle \in \omega \times A: \overline{p} \Vdash \dot{h}(n) = a\} \in M. \quad ∎$$

LEMMA 7: *Let $S \subseteq \lim(\omega_1)$ and T be an ω_1-tree which is S-properly pruned in countable products. Then for any index set I, the poset $\mathbb{P}(T, I, \omega_1)$ is ω_1-Baire.*

Proof: For each $n \in \omega$, let D_n be a dense open subset of $\mathbb{P}(T, I, \omega_1)$. Let $p \in \mathbb{P}(T, I, \omega_1)$. We now construct $p_s \in \mathbb{P}(T, I, \omega_1)$ for every $s \in 2^{<\omega}$ inductively on the length of s such that,

(1) $p_0 \leq p$,
(2) $s \subseteq t$ iff $p_t \leq p_s$,
(3) there is an increasing sequence $\langle \delta_n: n \in \omega\rangle$ of countable ordinals such that for every $s \in 2^n$, p_s is uniform at δ_n.
(4) for every $s \in 2^{<\omega}$, $p_{s^\frown\langle 0\rangle}$ and $p_{s^\frown\langle 1\rangle}$ are completely incompatible,
(5) for every $s \in 2^n$, $p_s \in D_n$.

Assume that we have already had p_s for every $s \in 2^{<n}$.

Let $s \in 2^{n-1}$ and $q^s \in D_n$ such that $q^s \leq p_s$. Let $l = 0, 1$. By Lemma 5, there are $q^s_l < q^s$ such that q^s_0 and q^s_1 are completely incompatible. Let

$$\delta_n = \bigcup\{\mathrm{ht}_T(q^s_l(i)) \colon i \in I, s \in 2^{n-1}, l = 0, 1\} + 1.$$

δ_n is countable because the support of every q^s_l is countable. Let $p_{s^\frown\langle l\rangle} \in \mathbb{P}(T, I, \omega_1)$ such that $p_{s^\frown\langle l\rangle} \leq q^s_l$ and all $p_{s^\frown\langle l\rangle}$ are uniform at δ_n. $p_{s^\frown\langle l\rangle} \in D_n$ because $p_{s^\frown\langle l\rangle} \leq q^s$.

Let $I' = \bigcup_{s \in 2^{<\omega}} \mathrm{supt}(p_s)$. Then I' is countable.

$$C \restriction I' = \{p_s \restriction I' \colon s \in 2^{<\omega}\}$$

is now a Cantor tree in $T^{I'}$, which is separated by (4) and uniformly cofinal in $(T \restriction \delta)^{I'}$, where $\delta = \bigcup_{n \in \omega} \delta_n$. Since T is S-properly pruned in countable products, there exists $f \in 2^\omega$ such that for every $i \in I'$, $\bigcup_{n\in\omega} p_{f\restriction n}(i) \in T_\delta \bigcup\{\emptyset\}$.

Let $p_f \in \mathbb{P}(T, I, \omega_1)$ defined by letting

$$p_f \restriction I' = \langle \bigcup_{n \in \omega} p_{f\restriction n}(i) \colon i \in I' \rangle$$

and

$$p_f \restriction I \setminus I' \equiv \emptyset.$$

Then $p_f \in \mathbb{P}(T, I, \omega_1)$ and $p_f \leq p_{f\restriction n}$ for every $n \in \omega$. So $p_f \leq p$ and $p_f \in \bigcap_{n\in\omega} D_n$. ∎

THEOREM 8: *Assume the existence of a strongly inaccessible cardinal. It is consistent with CH plus $2^{\omega_1} > \omega_2$ that there exists a Jech–Kunen tree and there are no Kurepa trees.*

Proof: Let M be a model of GCH plus that there is a strongly inaccessible cardinal κ. In M, let T be an ω_1-tree which is $\lim(\omega_1)$-properly pruned in countable products and let μ and λ be two regular cardinals such that $\kappa \leq \mu < \lambda$. Again in M let $\mathbb{P}_1 = Lv(\kappa, \omega_1)$, $\mathbb{P}_2 = \mathbb{P}(T, \mu, \omega_1)$ and $\mathbb{P}_3 = Fn(\lambda, 2, \omega_1)$. Let $G = G_1 \times G_2 \times G_3$ be a $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$-generic filter over M. We will show that $M[G]$ is a model of CH plus $\lambda = 2^{\omega_1} > \mu \geq \omega_2 = \kappa$, in which there are no Kurepa tree and T is a Jech–Kunen tree with μ-many branches.

CLAIM 8.1: $M^\omega \bigcap M[G] \subseteq M$.

Proof of Claim 8.1: We first force with $\mathbb{P}_2$. By Lemma 6 and Lemma 7, $\mathbb{P}_2$ is ω_1-Baire and forcing with $\mathbb{P}_2$ will not add any new countable sequences. Hence $\mathbb{P}_1 \times \mathbb{P}_3$ is still ω_1-closed in $M[G_2]$. Then forcing with $\mathbb{P}_1 \times \mathbb{P}_3$ will also not add any new countable sequences because it is ω_1-closed.

CLAIM 8.2: $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ *has the* κ*-c.c.*

Proof of Claim 8.2: Let

$$\{\langle p_\alpha, q_\alpha, r_\alpha\rangle : \alpha < \kappa\} \subseteq \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3.$$

By the Δ-system lemma, we can assume that the domains of all p_α, the domains of all q_α and the domains of all r_α form three Δ-systems with roots Δ_1, Δ_2 and Δ_3 respectively. Since there are less than κ-many p's in $\mathbb{P}_1$ with domains $= \Delta_1$, there are ω_1-many q's in $\mathbb{P}_2$ with domains $= \Delta_2$, and there are ω_1-many r's in $\mathbb{P}_3$ with domains $= \Delta_3$, then there exist α_1 and α_2 in κ such that

$$p_{\alpha_1} \restriction \Delta_1 = p_{\alpha_2} \restriction \Delta_1, \quad q_{\alpha_1} \restriction \Delta_2 = q_{\alpha_2} \restriction \Delta_2 \quad \text{and} \quad r_{\alpha_1} \restriction \Delta_3 = r_{\alpha_2} \restriction \Delta_3.$$

Obviously $\langle p_{\alpha_1}, q_{\alpha_1}, r_{\alpha_1}\rangle$ and $\langle p_{\alpha_2}, q_{\alpha_2}, r_{\alpha_2}\rangle$ are compatible.

Remark: By Claim 8.1 and Claim 8.2, ω_1 and all the cardinals greater than or equal to κ in M are preserved and CH is true in $M[G]$. In $M[G]$, $\kappa = \omega_2$ because forcing with $\mathbb{P}_1$ collapses all the cardinals between ω_1 and κ in M. Also in $M[G]$, $2^{\omega_1} = \lambda$ because forcing with $\mathbb{P}_3$ adds λ-many subsets of ω_1.

CLAIM 8.3: *There are no Kurepa trees in* $M[G]$.

Proof of Claim 8.3: Suppose that is not true. Let K be a normal Kurepa tree in $M[G]$. Since $|K| = \omega_1$, there are $\theta < \kappa$, $I \subseteq \mu$ with $|I| \leq \omega_1$ and $J \subseteq \lambda$ with $|J| \leq \omega_1$ such that

$$K \in M[G'] = M[G_1' \times G_2' \times G_3'],$$

where

$$G_1' = G_1 \bigcap Lv(\theta, \omega_1),$$

$$G_2' = G_2 \bigcap \mathbb{P}(T, I, \omega_1),$$

$$G_3' = G_3 \bigcap Fn(J, 2, \omega_1)$$

and

$$G' = G_1' \times G_2' \times G_3'.$$

Let

$$G_1'' = G_1 \bigcap Lv(\kappa \setminus \theta, \omega_1),$$

$$G_2'' = G_2 \bigcap \mathbb{P}(T, \mu \setminus I, \omega_1),$$

$$G_3'' = G_3 \bigcap Fn(\lambda \setminus J, 2, \omega_1)$$

and

$$G'' = G_1'' \times G_2'' \times G_3''.$$

Since $M[G'] \models 2^{\omega_1} < \kappa$, there exists

$$b \in \mathcal{B}(K) \bigcap M[G] \setminus M[G'].$$

Furthermore

$$b \notin M[G'][G_1''][G_3'']$$

because $Lv(\kappa \setminus \theta, \omega_1)$ and $Fn(\lambda \setminus J, 2, \omega_1)$ are ω_1-closed in $M[G']$. We now work in $M[G'][G_1''][G_3'']$ and let $p \in G_2''$ such that

$$p \Vdash (\dot{b} \in \mathcal{B}(K) \bigcap M[\dot{G}] \setminus M[G'][G_1''][G_3'']).$$

We construct

$$C = \{p_s \colon s \in 2^{<\omega}\} \subseteq \mathbb{P}(T, \mu \setminus I, \omega_1)$$

and

$$D = \{k_s \colon s \in 2^{<\omega}\} \subseteq K$$

such that,

(1) $s \subsetneq s'$ iff $p_{s'} \leq p_s$ iff $k_s \leq k_{s'}$,

(2) C is separated and uniformly cofinal in $(T \restriction \delta)^{\mu \setminus I}$ for some $\delta \in \lim(\omega_1)$,

(3) D is cofinal in $K \restriction \delta'$ for some $\delta' \in \lim(\omega_1)$,

(4) for every $s \in 2^{<\omega}$, $p_s \Vdash k_s \in \dot{b}$.

Assume that we have already had p_s and k_s for all $s \in 2^{<n}$. Let

$$\delta_n' = \bigcup\{\mathrm{ht}_K(k_s) \colon s \in 2^{<n}\} + 1$$

and pick $s \in 2^{n-1}$. Let $l = 0, 1$.

First find $p'_s \leq p_s$ such that

$$\exists x \in K_{\delta'_n}(p'_s \Vdash x \in \dot{b}).$$

Since

$$p'_s \Vdash \dot{b} \notin M[G'][G''_1][G''_3],$$

there exist $q^s_l \leq p'_s$ and $x_l > x > k_s$ such that $x_0 \perp x_1$ and $q^s_l \Vdash x_l \in \dot{b}$. By Lemma 5, we can extend q^s_l to r^s_l such that r^s_l are uniform at $\alpha_s < \omega_1$ and r^s_0 is completely incompatible with r^s_1. Let

$$\delta_n = \bigcup\{\alpha_s : s \in 2^{n-1}\} + 1,$$

$p_{s^\frown\langle l\rangle}$ be an extension of r^s_l such that $\mathrm{supt}(p_{s^\frown\langle l\rangle}) = \mathrm{supt}(r^s_l)$ and $p_{s^\frown\langle l\rangle}$ be uniform at δ_n. This ends the construction.

Let $\delta = \bigcup_{n\in\omega}\delta_n$, $\delta' = \bigcup_{n\in\omega}\delta'_n$ and $I' = \bigcup_{s\in 2^{<\omega}}\mathrm{supt}(p_s)$. Then I' is countable. Since T is $\lim(\omega_1)$-properly pruned in countable products and $C \restriction I'$ is a Cantor tree which is separated and uniformly cofinal in $(T\restriction\delta)^{I'}$, then there are uncountably many $f \in 2^\omega$ such that p_f defined by letting

$$p_f(i) = \bigcup_{n\in\omega} p_{f\restriction n}(i)$$

for every $i \in I'$ is a lower bound of $\{p_{f\restriction n} : n \in \omega\}$ in $\mathbb{P}(T, \mu\setminus, \omega_1)$. (Note that C is in M because no new countable sequences are added.) For every such f there exists $k_f \in K_{\delta'}$ such that $p_f \Vdash k_f \in \dot{b}$ and for different f, k_f are different. That contradicts that K is a Kurepa tree.

CLAIM 8.4: $M[G] \models (|\mathcal{B}(T)| = \mu)$.

Proof of Claim 8.4: $|\mathcal{B}(T)| \geq \mu$ is trivial because forcing with $\mathbb{P}_2$ adds at least μ-many new branches of T. Since in $M[G_1][G_2]$, $2^{\omega_1} = \mu$, then we need only to show that forcing with $\mathbb{P}_3$ will not add any new branches of T.

Suppose that is not true and let b be a branch of T, which is in $M[G]$ $M[G_1][G_2]$. We now work in $M[G_1][G_2]$ and let $p \in G_3$ such that

$$p \Vdash \dot{b} \in \mathcal{B}(T) \bigcap M[\dot{G}] \setminus M[G_1][G_2].$$

We can then easily construct $C = \{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}_3$ and $D = \{t_s : s \in 2^{<\omega}\} \subseteq T$ such that

(1) $s \subseteq s'$ iff $p_{s'} \leq p_s$ iff $t_s \leq t_{s'}$,

(2) D is a Cantor tree which is cofinal in $T \restriction \delta$ for some $\delta \in \lim(\omega_1)$,

(3) for every $s \in 2^{<\omega}$, $p_s \Vdash t_s \in \dot{b}$.

Since T is $\lim(\omega_1)$-properly pruned by Lemma 3, there exists $g \in 2^\omega$ such that $\bigcup_{n \in \omega} t_{g \restriction n} \notin T_\delta$. But $\mathbb{P}_3$ is ω_1-closed in $M[G_1][G_2]$ because no new countable sequences have been added. Hence there exists $p_g \in \mathbb{P}_3$ such that $p_g \leq p_{g \restriction n}$ for every $n \in \omega$. This implies that there exists $t \in T_\delta$ such that $p_f \Vdash t \in \dot{b}$. Hence

$$t = \bigcup_{n \in \omega} t_{g \restriction n} \in T_\delta,$$

a contradiction. ∎

In the model constructed above, we forced only one Jech–Kunen tree. Next we will build a model of *CH* plus $2^{\omega_1} > \omega_2$, in which there are no Kurepa trees and there are many Jech–Kunen trees with different numbers of branches.

THEOREM 9: *Assume the existence of a strongly inaccessible cardinal. It is consistent with CH plus $2^{\omega_1} > \omega_2$ that there are no Kurepa trees and there are Jech–Kunen trees T^α for $\alpha \in \omega_1$ such that $\alpha \neq \alpha'$ implies $|\mathcal{B}(T^\alpha)| \neq |\mathcal{B}(T^{\alpha'})|$.*

Proof: Let M be a model of GCH and that there exists a strongly inaccessible cardinal κ. In M, let

$$\Gamma = \{\mu_\alpha : \alpha \in \omega_1\} \subseteq [\kappa, \lambda)$$

be a set of different regular cardinals, where λ is also a regular cardinal. Again in M, let $\{S_\alpha : \alpha \in \omega_1\}$ be a partition of $\lim(\omega_1)$ such that every S_α is a stationary, and let T^α be an ω_1-tree which is S_α-properly pruned in countable products for every $\alpha \in \omega_1$. In M, let $\mathbb{P}_1 = Lv(\kappa, \omega_1)$, $\mathbb{P}_2$ be the product of $\{\mathbb{P}(T^\alpha, \mu_\alpha, \omega_1) : \alpha \in \omega_1\}$ with countable supports, and $\mathbb{P}_3 = Fn(\lambda, 2, \omega_1)$. Let $G = G_1 \times G_2 \times G_3$ be a $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$-generic filter over M. Then $M[G]$ is the model we are looking for.

CLAIM 9.1: $M^\omega \bigcap M[G] \subseteq M$.

CLAIM 9.2: $\mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3$ *has the κ-c.c.*

CLAIM 9.3: *There are no Kurepa trees in $M[G]$.*

All the proofs of above three claims are similar to the proofs of corresponding claims in Theorem 8. By Claim 9.1 and Claim 9.2, ω_1 and all the cardinals greater than or equal to κ are preserved. Besides, forcing with $\mathbb{P}_1$ collapses all the cardinals between ω_1 and κ. So in $M[G]$, *CH* is true, $\kappa = \omega_2 < \lambda = 2^{\omega_1}$ and $\{\mu_\alpha : \alpha \in \omega_1\} \subseteq [\kappa, \lambda)$ is still a set of different cardinals.

CLAIM 9.4: *$M[G] \models (|\mathcal{B}(T^\alpha)| = \mu_\alpha)$ for every $\alpha \in \omega_1$.*

Proof of Claim 9.4: Pick an $\alpha \in \omega_1$. Let $\mathbb{P}_2^\alpha = \mathbb{P}(T^\alpha, \mu_\alpha, \omega_1)$ and $\mathbb{P}_2^{-\alpha}$ be the product of $\{\mathbb{P}(T^\beta, \mu_\beta, \omega_1) : \beta \in \omega_1 \setminus \{\alpha\}\}$ with countable supports. Then $\mathbb{P}_2 \cong \mathbb{P}_2^\alpha \times \mathbb{P}_2^{-\alpha}$. Let $p \in \mathbb{P}_2$. We let

$$\mathrm{SUPT}(p) = \{\alpha \in \omega_1 : \mathrm{supt}(p(\alpha)) \neq \emptyset\}.$$

Notice the differences between supt and SUPT. We call an element $p \in \mathbb{P}_2^{-\alpha}$ uniform at γ for some $\gamma \in \omega_1$ if for every $\beta \in \mathrm{SUPT}(p)$, $p(\beta)$ is uniform at γ.

SUBCLAIM 9.4.1: *Forcing with $\mathbb{P}_2^{-\alpha}$ will not add any new branches to T^α.*

Proof of Subclaim 9.4.1: Let $G_2 = G_2^\alpha \times G_2^{-\alpha} \subseteq \mathbb{P}_2^\alpha \times \mathbb{P}_2^{-\alpha}$. Suppose that Subclaim 1 is not true and let b be a branch of T^α such that

$$b \in M[G_1][G_2] \setminus M[G_1][G_2^\alpha].$$

We now work in $M[G_1][G_2^\alpha]$ and let $p \in G_2^{-\alpha}$ such that

$$p \Vdash \dot{b} \in \mathcal{B}(T^\alpha) \setminus M[G_1][G_2^\alpha].$$

We construct recursively a normal subtree T' of T^α with every level countable, and a subset $C = \{p_t : t \in T'\}$ of $\mathbb{P}_2^{-\alpha}$ such that,

(1) for every $\delta \in \omega_1$ there is γ_δ such that $T'_\delta \subseteq T^\alpha_{\gamma_\delta}$,
(2) if $\delta \in \lim(\omega_1)$, then $\gamma_\delta = \bigcup_{\beta<\delta} \gamma_\beta$,
(3) $p_\emptyset \leq p$, and for any $t, t' \in T'$, $t \leq t'$ iff $p_{t'} \leq p_t$,
(4) for every $t \in T'_\delta$, there is γ', $\gamma_\delta \leq \gamma' \leq \gamma_{\delta+1}$ such that p_t is uniform at γ',
(5) if $t \in T'_\delta$ for some $\delta \in \lim(\omega_1)$, then p_t is uniform at $\mathrm{ht}_{T^\alpha}(t) = \gamma_\delta$,
(6) $t \perp t'$ implies that $p_t(\beta)$ and $p_{t'}(\beta)$ are completely incompatible for every $\beta \in \mathrm{SUPT}(p_t) \bigcap \mathrm{SUPT}(p_{t'})$,
(7) for every $t \in T'$, $p_t \Vdash t \in \dot{b}$.

Assume that we have already had $T' \restriction \delta$ and $C \restriction \delta = \{p_t : t \in T' \restriction \delta\}$.

CASE 1: *$\delta = \gamma + 1$ for some $\gamma \in \omega_1$.* Pick $t \in T'_\gamma$ and let $l = 0, 1$.

Since

$$p_t \Vdash \dot{b} \notin M[G_1][G_2^\alpha],$$

there exist $t_l \in T^\alpha, t_l > t$ and $q_l^t \leq p_t$ such that

$$t_0 \perp t_1 \quad \text{and} \quad q_l^t \Vdash t_l \in \dot{b}.$$

Without loss of generality we can pick t_l such that $\mathrm{ht}_{T^\alpha}(t_l) = \delta'$ for every $t \in T'_\gamma$ and $l = 0, 1$, where

$$\delta' > \bigcup\{\gamma'' : p_t \text{ is uniform at } \gamma'' \text{ for some } t \in T'_\gamma\}.$$

Besides, we can require that $q_0^t(\beta)$ and $q_1^t(\beta)$ are uniform and are completely incompatible for every $t \in T'_\gamma$ and $\beta \in \mathrm{SUPT}(q_0^t) \bigcap \mathrm{SUPT}(q_1^t)$. Let $\gamma' \in \omega_1$ such that $\gamma' > \delta'$ and

$$\gamma' > \bigcup\{\gamma'' : q_l^t \text{ is uniform at } \gamma'' \text{ for some } t \in T'_\gamma \text{ and } l = 0, 1\}.$$

Let $T'_\delta = \{t_l : t \in T'_\gamma, l = 0, 1\}$ and let $p_{t_l} \leq q_l^t$ such that p_{t_l} is uniform at γ'.

Case 2: $\delta \in \lim(\omega_1)$. First γ_δ can't be in S_α because otherwise every T^β for $\beta \in \omega_1 \setminus \{\alpha\}$ is complete at level γ_δ. But in $M[G_1][G_2^\alpha]$, T^α is still properly pruned at level γ_δ because forcing with $\mathbb{P}_1 \times \mathbb{P}_2^\alpha$ adds no new countable sequences, so that there exists $B \in \mathcal{B}(T' \restriction \delta)$ such that B has no upper bound in T^α. On the other hand, $\{p_t : t \in B\}$ has a lower bound p_B in $\mathbb{P}_2^{-\alpha}$. Then

$$p_B \Vdash \exists t \in T^\alpha_{\gamma_\delta}(t \in \dot{b})$$

implies that B has an upper bound in T^α, a contradiction.

Assume that $\gamma_\delta \in S_\beta$ for some $\beta \neq \alpha$. Since in $M[G_1][G_2^\alpha]$, T^β is properly pruned at level γ_δ, then for every $t \in T' \restriction \delta$ there exists $B_t \in \mathcal{B}(T' \restriction \delta)$ such that $t \in B_t$ and $\langle \bigcup_{t' \in B_t} p_{t'}(\beta)(i) \rangle_{i \in \mu_\beta} \in \mathbb{P}(T^\beta, \mu_\beta, \omega_1)$.

Now every $T^{\beta'}$ is complete at level γ_δ for $\beta' \neq \beta$. We can define $p_{B_t} \in \mathbb{P}_2^{-\alpha}$ by letting

$$p_{B_t}(\beta)(i) = \bigcup_{t' \in B_t} p_{t'}(\beta)(i)$$

for every $\beta \in \omega_1 \setminus \{\alpha\}$ and $i \in \mu_\beta$.

Let $T'_\delta = \{\bigcup B_t : t \in T' \restriction \delta\}$ and let $p_{\bigcup B_t} = p_{B_t}$. This ends the construction.

Since S_α is stationary and by (2), $\{\gamma_\delta : \delta \in \omega_1\}$ is a club set, then there exists $\delta \in \omega_1$ such that $\gamma_\delta \in S_\alpha$. But this has been shown impossible.

SUBCLAIM 9.4.2: *Forcing with* $\mathbb{P}_3$ *will not add any new branches to* T^α.

Proof of Subclaim 9.4.2: Similar (but much easier) to the proof of Subclaim 9.4.1.

By Subclaim 9.4.1 and Subclaim 9.4.2, all the branches of T^α in $M[G]$ are already in $M[G_1][G_2^\alpha]$. But in $M[G_1][G_2^\alpha]$ $2^{\omega_1} = \mu_\alpha$. So $|\mathcal{B}(T^\alpha)| = \mu_\alpha$. ∎

Concluding remarks: (1) μ, μ_α and λ are not necessarily regular.

(2) In Theorem 9, we can also have larger number of trees. For this we use S_α's which are only almost disjoint.

(3) In the proof of Theorem 9, if we do not want to use stationary sets, we can force the trees as part of the forcing, and then prove that they are "pruned together", so using the stationary sets simplifies the matter.

(4) We have used $(\lim(\omega_1) \setminus S)$–complete tree T (*i. e.* every branch of $T \restriction \delta$ for $\delta \in \lim(\omega_1) \setminus S$ has an upper bound in T). Our consideration leads naturally to S-Kurepa trees. T is called an S-Kurepa tree if $T = \bigcup_{\alpha\in\omega_1} T(\alpha)$, where $\mathrm{ht}(T(\alpha)) = \alpha$, $T(\alpha) = \bigcup_{\beta<\alpha} T(\beta)$ if $\alpha \in \lim(\omega_1)$ and $|T_\alpha \bigcap \{\bigcup B \colon B \in \mathcal{B}(T(\alpha))\}| \leq \omega$ if $\alpha \in S$. So we may well consider S-Kurepa and $(\lim(\omega_1) \setminus S)$-complete trees.

(5) The T we build are not only $(\lim(\omega_1) \setminus S)$-complete, but also strongly proper (see [S1] or/and [S2]).

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